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SOME NEW MODELS OF THE TIME-FRACTIONAL GAS DYNAMICS EQUATION

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Abstract. In this paper we extend and investigate the model of the gas dynamic equation (GDE) to some new models involving the time-fractional gas dynamic equation (TFGDE) with the Liouville-Caputo (LC), Caputo-Fabrizio (CFC) and Atangana-Baleanu (ABC) time-fractional derivatives. We employ the Homotopy Analysis Transform Method (HATM) to calculate the approximate solutions of TFGDE by using LC, CFC and ABC in the Liouville-Caputo sense. We study the convergence analysis of HATM by finding the interval of convergence through the h-curves. We also show the effectiveness and accuracy of this method by comparing the approximate solutions based upon the LC, CFC and ABC time – fractional derivatives.

Keywords: *Time-fractional gas dynamics equation (TFGDE), Homotopy Analysis Transform Method (HATM), Liouville-Caputo (LC), Caputo-Fabrizio (CFC) and Atangana-Baleanu (ABC) time-fractional derivatives.*

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1. Introduction

In this paper, we consider the Homotopy Analysis Transform Method (HATM) based upon the Liouville-Caputo (LC), Caputo-Fabrizio (CFC) and Atangana-Baleanu (ABC) time-fractional derivatives. It is applied here to find the solution of the homogeneous time-fractional gas dynamics equation (TFGDE) given by

$$\frac{\partial^\alpha \psi}{\partial \tau^\alpha} + \psi \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0, \tag{1}$$

where

$$(\varsigma,\tau) \in (0,\infty)\times(0,\tau_0) \text{ and } 0<\alpha\leq 1. \tag{2}$$

Here, and in what follows, α, $\psi(\varsigma,\tau)$, ς and τ represent the order of the fractional derivative, the probability density function, the time coordinate and the spectral coordinate, respectively.

The classical gas dynamics equation is obtained by putting $\alpha = 1$ in (1). The essentials of the gas dynamic equation are based upon the physical laws of conservation, namely, the laws of conservation of mass, momentum and energy. Some fractional-derivative models were considered in the earlier works (see (Das & Kumar, 2011; Hemida & Mohamed, 2010; Saad, 2018; Saad & Al-Shomrani, 2016).

In the past several decades, various real-world issues have been modeled in many areas by using some very powerful tools. One of these tools is fractional calculus. Several important definitions have been introduced for of fractional-order derivatives, including: the Riemann-Liouville, the Grunvald-Letnikov, the Liouville-Caputo, the Caputo-Fabrizio and the Atangana-Baleanu fractional-order derivatives (see, for example, Atangana & Baleanu, 2016; Caputo & Fabrizio, 2015; Cattani *et al.*,2015; Kilbas *et al.*,2006; Ma *et al.*,2016; Podlubny, 1999).

Using the fundamental relations of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative was constructed, which involves the convolution of a given function and a power-law kernel (see,for details, (Kilbas *et al.*,2006; Podlubny, 1999). The Liouville-Caputo fractional derivative involves the convolution of the local derivative of a given function with a power –low function [12]. Recently, Caputo and Fabrizio (Caputo & Fabrizio, 2015) and Atangana and Beleanu (Atangana & Baleanu, 2016) proposed some interesting fractional –order derivatives based upon the exponential decay law which is a generalized power-law function (Alsaedi *et al.*,2016; Atangana, 2016; Atangana & Alkahtini, 2015a, 2015b; 2016; Atangana & Nieto, 2015). The Caputo-Fabrizio fractional- order derivative as well as the Atangana-Beleanu derivative allow us to describle complex physical problems that follow, at the same time, the power law and the exponential decay law (see (Alsaedi *et al.*, 2016; Atangana, 2016; Atangana & Alkahtini, 2015a, 2015b; 2016; Atangana & Nieto, 2015)).

The main goal of the paper is to obtain the approximate solutions to TFGDE by applying the above-mentioned LC, CFC and ABC operators and HATM. The present paper is organized as follows: The next section (Section 2) is devoted to compute the HATM solutions by using the LC, CFC and ABC operators. In Section 3, we give several graphical representations as well as numerical results and consider their efficiencies and effectiveness. In the last section (Section 4), we present our concluding remarks and observations.

2. New HATM Solutions Based Upon the LC, CFC and ABS operators

In this section, we use the HATM| (see, for example (Kumar *et al.*, 2017; Saad & Al-Shomrani, 2016) in order to solve the LC, CFC and ABC analogues of the TFGDE (1). To obtain these analogues equations, we replace the time-fractional derivative $\frac{\partial^\alpha \psi}{\partial \tau^\alpha}$ in the TFGDE (1) by ${}^{LC}_{0}D^{\alpha}_{\tau}\psi$, ${}^{CFC}_{0}D^{\alpha}_{\tau}\psi$ and ${}^{ABC}_{0}D^{\alpha}_{\tau}\psi$, successively, where the order α of the time-fractional derivatives is constrained by

$$n-1 < \alpha \le n, \quad (n \in N := \{1,2,3,...\}).$$

The corresponding LC, CFC and ABC time-fractional analogues of the TFGDE (1) are given by

$${}^{LC}_{0}D^{\alpha}_{\tau}\psi + \psi \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \quad (0 < \alpha \le 1;\ \varsigma \in R;\ \tau > 0), \tag{3}$$

$$ {}_{0}^{CFC}D_{\tau}^{\alpha}\psi + \psi \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \quad (0 < \alpha \leq 1;\ \varsigma \in R;\ \tau > 0), \tag{4} $$

$$ {}_{0}^{ABC}D_{\tau}^{\alpha}\psi + \psi \frac{\partial \psi}{\partial \varsigma} - \psi(1-\psi) = 0 \ (0 < \alpha \leq 1;\ \varsigma \in R;\ \tau > 0), \tag{5} $$

respectively. Here ${}_{0}^{LC}D_{\tau}^{\alpha}\psi$ and ${}_{0}^{CFC}D_{\tau}^{\alpha}\psi$ denote the time-fractional derivatives of order α for a suitably defined function $f(\tau)$, which are defined, respectively, by

$$ {}_{0}^{LC}D_{\tau}^{\alpha}(f(\tau)) = J^{m-\alpha}D^{m}(f(\tau)) = \frac{1}{\Gamma(m-\alpha)}\int_{0}^{\tau}(\tau-t)^{m-\alpha-1} f^{(m)}(t)dt $$

$$ (m-1 < \alpha \leq m : m \in N;\ f \in C_{\mu}^{m};\ \mu \geq -1) $$

and

$$ {}_{0}^{CFC}D_{\tau}^{\alpha}(f(\tau)) = \frac{M(\alpha)}{1-\alpha}\int_{0}^{\tau} \exp\left(-\frac{\alpha(\tau-t)}{1-\alpha}\right) D(f(t))dt $$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$ and ${}_{0}^{ABC}D_{\tau}^{\alpha}(f(\tau))$ is known as the ABC time-fractional derivative of order α in the Liouville-Caputo sense given, for a suitably defined function $f(\tau)$, by

$$ {}_{0}^{ABC}D_{\tau}^{\alpha}(f(\tau)) = \frac{M(\alpha)}{1-\alpha}\int_{0}^{\tau} E_{\alpha}\left(-\frac{\alpha(\tau-t)}{1-\alpha}\right) D(f(t))dt, $$

where

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(ak+1)} $$

is the Mittag-Leffter function and $M(\alpha)$ is a normalized with the same properties as in the Liouville –Caputo (LC) and the Caputo-Fabrizio (CFC) cases.

If $0 < \alpha \leq 1$, then we define the Laplace of the Liouville-Caputo (LC), the Caputo-Fabrizio (CFC) and the Atangana-Beleanu (ABC) fractional-order derivatives as follows (see, for example (Atangana & Baleanu, 2016; Atangana & Koca, 2016; Caputo & Fabrizio, 2015, 2016; Losada & Nieto, 2015; Saad, 2018):

$$ L\left[{}_{0}^{LC}D_{\tau}^{\alpha}\psi(\varsigma,\tau)\right] = s^{\alpha}L[\psi(\varsigma,\tau)] - \sum_{k=0}^{m-1}\psi^{(k)}(\varsigma,0+)s^{\alpha-k-1}, \tag{6} $$

$$ L\left[{}_{0}^{CFC}D_{\tau}^{\alpha}\psi(\varsigma,\tau)\right] = M(\alpha)\left(\frac{s^{\alpha}L[\psi(\varsigma,\tau)]\ (s) - \psi(\varsigma,0)}{\alpha+(1-\alpha)s}\right), \tag{7} $$

and

$$ L\left[{}_{0}^{ABC}D_{\tau}^{\alpha}\psi(\varsigma,\tau)\right] = M(\alpha)\left(\frac{s^{\alpha}L[\psi(\varsigma,\tau)]\ s^{\alpha-1} - \psi(\varsigma,0)}{\alpha+(1-\alpha)s^{\alpha}}\right), \tag{8} $$

respectively. The initial condition is taken by letting $\tau = 0$ in the exact solution

$$ \psi(\varsigma,\tau) = 1 - e^{\tau-\varsigma} $$

and we thus have

$$ \psi(\varsigma,0) = 1 - e^{-\varsigma}. \tag{9} $$

By applying the Laplace transform to the equations (3) to (5) and using the Laplace transform formulas (6) to (8), we obtain

$$s^{\alpha} L[\psi(\varsigma,\tau)] - \frac{\psi(\varsigma,0)}{s^{1-\alpha}} + L[\psi\psi_{\varsigma} - \psi(1-\psi)] = 0, \tag{10}$$

$$M(\alpha)\left(\frac{sL[\psi(\varsigma,\tau)] - \psi(\varsigma,0)}{\alpha + (1-\alpha)s}\right) + L[\psi\psi_{\varsigma} - \psi(1-\psi)] = 0, \tag{11}$$

and

$$M(\alpha)\left(\frac{s^{\alpha} L[\psi(\varsigma,\tau)] - s^{\alpha-1}\psi(\varsigma,0)}{\alpha + (1-\alpha)s^{\alpha}}\right) + L[\psi\psi_{\varsigma} - \psi(1-\psi)] = 0, \tag{12}$$

respectively.
Upon simplifying these last equations (10) to (12), we find that

$$L[\psi(\varsigma,\tau)] - \frac{\psi(\varsigma,0)}{s} + \Omega(\cdot)\, L[\psi\psi_{\varsigma} - \psi(1-\psi)] = 0, \tag{13}$$

where

$$\Omega(LC) = \frac{1}{s^{\alpha}},$$

$$\Omega(CFC) = \frac{\alpha + (1-\alpha)s}{sM(\alpha)},$$

and

$$\Omega(ABC) = \frac{1 + \alpha(s^{\alpha} - 1)}{M(\alpha)}.$$

We now define the nonlinear operator N as follows (see (Kumar *et al.*,2017; Saad *et al.*, 2017; Saad & Al-Shomrani, 2016; Singh *et al.*,2013; Srivastava *et al.*,2017):

$$\begin{aligned} N[\psi(\varsigma,\tau;q)] &= L[\psi(\varsigma,\tau;q)] - \frac{\psi(\varsigma,0)}{s}(1-\aleph_m) \\ &+ \Omega(\cdot)L[\psi(\varsigma,\tau,q)\psi_s(\varsigma,\tau;q) - \varphi(\varsigma,\tau;q)(1-\varphi(\varsigma,\tau;q))], \end{aligned} \tag{14}$$

where $q \in [0,1]$ is an embedding parameter and $\varphi(\varsigma,\tau;q)$ is a real function of ς,τ and q. Liao (Liao, 1992; 2003; 2004; 2005) constructed the following zero-order deformation equation:

$$(1-q)L[\varphi(\varsigma,\tau;q) - \psi_0(\varsigma,\tau)] = qhH(\varsigma,\tau)N[\psi(\varsigma,\tau;q)], \tag{15}$$

where $h \neq 0$ and $H(\varsigma,\tau) \neq 0$ are an auxiliary parameter and an auxiliary function, respectively, $\psi_0(\varsigma,\tau)$ is an initial guess for $\psi(\varsigma,\tau)$ and $\varphi(\varsigma,\tau;q)$ is an unknown function.

Obviously, when $q = 0$ and $q = 1$, we have

$$\varphi(\varsigma,\tau;0) = \psi_0(\varsigma,\tau) \text{ and } \varphi(\varsigma,\tau;1) = \psi(\varsigma,\tau), \tag{16}$$

respectively. Thus, as q increases from 0 to 1, the solution $\varphi(\varsigma,\tau;q)$ varies from the initial guess $\psi_0(\varsigma,\tau)$ to the solution $\psi(\varsigma,\tau)$. Upon expanding $\varphi(\varsigma,\tau;q)$ in a Taylor series with respect to q, we have

$$\varphi(\varsigma,\tau;q) = \psi_0(\varsigma,\tau) + \sum_{m=1}^{\infty} \psi_m(\varsigma,\tau) q^m, \tag{17}$$

where

$$\psi_m(\varsigma,\tau) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}(\varphi(\varsigma,\tau;q))\bigg|_{q=0}. \tag{18}$$

If $L, \psi_0(\varsigma,\tau)$, h and $H(\varsigma,\tau)$ are properly chosen, the series in (17) converges at $q=1$ and we have

$$\psi(\varsigma,\tau)=\psi_0(\varsigma,\tau)+\sum_{m=1}^{\infty}\psi_m(\varsigma,\tau). \tag{19}$$

Let us now define the vectors $\vec{\psi}_m(\varsigma,\tau)$ by

$$\vec{\psi}_m(\varsigma,\tau)=\{\psi_0(\varsigma,\tau),\psi_1(\varsigma,\tau),\psi_2(\varsigma,\tau),\ldots,\psi_m(\varsigma,\tau)\}. \tag{20}$$

Upon differentiating both sides of the equation (15) times with respect to q, if we set $q=0$ and then divide them by $m!$, we have the so-called m-th order deformation equation:

$$L[\vec{\psi}_m(\varsigma,\tau)-\aleph_m\psi_{m-1}(\varsigma,\tau)]=hH(\varsigma,\tau)R_m(\vec{\psi}_{m-1}(\varsigma,\tau)). \tag{21}$$

Here

$$R_m(\vec{\psi}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(N[\phi(\varsigma,\tau;q))\bigg|_{q=0} \tag{22}$$

and

$$\aleph_m=\begin{cases}0 & (m\leq 1)\\ 1 & (m>1).\end{cases}$$

Upon computing the inverse Laplace transform of each member of the equation (21), we find the following power series solution:

$$\psi(\varsigma,\tau)=\sum_{m=0}^{\infty}\psi_m(\varsigma,\tau).$$

In view of (14) and (22), for $m\geq 1$ the solution of the m-th order deformation equation (21) is given by

$$\psi_m(\varsigma,\tau)-\aleph_m\psi_{m-1}(\varsigma,\tau)+hH(\varsigma,\tau)L^{-1}[\Re_m(\vec{\psi}_{m-1}(\varsigma,\tau)], \tag{23}$$

where

$$\Re_m(\vec{\psi}_{m-1})=L[(\vec{\psi}_{m-1}]-\frac{\psi(\varsigma,0)}{s}(1-\aleph_m)+\Omega(\cdot)L\left[\sum_{j=0}^{m=1}\psi_j(\psi_{m-1-j})_\varsigma-\psi_{m-1}(1-\psi_{m-1-j})\right]. \tag{24}$$

Consequently, the first there terms of the HATM approximate series solution of the TFGDEs (3) to (5) are given

$$\psi_0^{(\cdot)}(\varsigma,\tau)=1-e^{-\varsigma}, \tag{25}$$

$$\psi_1^{LC}(\varsigma,\tau)=\frac{ahe^{-\varsigma}\tau^{\alpha}}{\Gamma(\alpha+1)}, \tag{26}$$

$$\psi_2^{LC}(\varsigma,\tau)=\psi_1^{LC}(\varsigma,\tau)+ahe^{-\varsigma}\tau^{\alpha}\left(\frac{\tau^{\alpha}}{\Gamma(2\alpha+1)}-\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right), \tag{27}$$

$$\psi_3^{LC}(\varsigma,\tau)=\psi_2^{LC}(\varsigma,\tau)-ah^2e^{-\varsigma}\tau^{\alpha}\left[\frac{h+1}{\Gamma(\alpha+1)}+\tau^{\alpha}\left(\frac{h\tau^{\alpha}}{\Gamma(3\alpha+1)}-\frac{2h+1}{\Gamma(2\alpha+1)}\right)\right], \tag{28}$$

$$\psi_1^{CF}(\varsigma,\tau)=-\frac{ahe^{-\varsigma}[1-\alpha(1-\tau^{\alpha})]}{M(\alpha)}, \tag{29}$$

$$\psi_2^{CF}(\varsigma,\tau)=\psi_2^{CF}(\varsigma,\tau)+\frac{ah^2e^{-\varsigma}(-2\alpha[M(\alpha)-2](\tau-1)-2M(\alpha)+\alpha^2[\tau(\tau-4)+2]+2)}{2[M(\alpha)]^2}, \tag{30}$$

and

$$\psi_3^{CF}(\varsigma,\tau)=\psi_2^{CF}(\varsigma,\tau)+h[\psi_{31}^{CF}(\varsigma,\tau)+\psi_{32}^{CF}(\varsigma,\tau)+\psi_{33}^{CF}(\varsigma,\tau)], \tag{31}$$

where

$$\psi_{31}^{CF}(\varsigma,\tau)=\frac{a\alpha^2h\tau^2e^{-\varsigma}[3\alpha h+2hM(\alpha)-3h+M(\alpha)]}{2[M(\alpha)]^3}-\frac{a\alpha^3h^2\tau^3e^{-\varsigma}}{6[M(\alpha)]^3},$$

$$\psi_{32}^{CF}(\varsigma,\tau)=\frac{a(\alpha-1)he^{-\varsigma}[\alpha+M(\alpha)--1][\alpha h+hM(\alpha)-h+M(\alpha)]}{[M(\alpha)]^3},$$

$$\psi_{33}^{CF}(\varsigma,\tau)=\frac{a\alpha h\tau e^{-\varsigma}[h[\alpha+M(\alpha)-1][3\alpha+M(\alpha)-3]+M(\alpha)[2\alpha+M(\alpha)-2]}{[M(\alpha)]^3},$$

$$\psi_1^{ABC}(\varsigma,\tau)=-\frac{ahe^{-\varsigma}}{M(\alpha)}\left(1-\alpha+\frac{a\tau^\alpha}{\Gamma(\alpha+1)}\right), \tag{32}$$

$$\psi_2^{ABC}(\varsigma,\tau)=\psi_1^{ABC}(\varsigma,\tau)+\frac{ah^2e^{-\varsigma}}{[M(\alpha)]^2\Gamma(\alpha+1)\Gamma(2\alpha+1)}\psi_{21}^{ABC}(\varsigma,\tau), \tag{33}$$

$$\begin{aligned}&\psi_{21}^{ABC}(\varsigma,\tau)=\Gamma(2\alpha+1)((\alpha-1)\Gamma(\alpha+1)\Gamma(\alpha+1)[\alpha+M(\alpha)-1]-\\&-\alpha[2\alpha+M(\alpha)-2]\tau^\alpha)+\alpha^2\Gamma(\alpha)\tau^{2\alpha}\end{aligned} \tag{34}$$

$$\begin{aligned}&\psi_3^{ABC}(\varsigma,\tau)=\psi_2^{ABC}(\varsigma,\tau)+\frac{ae^{-\varsigma}h^2}{[M(\alpha)]^3\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\\&\times[\psi_{31}^{ABC}(\varsigma,\tau)+\psi_{32}^{ABC}(\varsigma,\tau)+\psi_{33}^{ABC}(\varsigma,\tau)],\end{aligned} \tag{35}$$

$$\psi_{31}^{ABC}(\varsigma,\tau)=\alpha\tau^2(h[\alpha+M(\alpha)-1][3\alpha+M(\alpha)-3]+M(\alpha)[2\alpha+M(\alpha)-2]),$$

$$\psi_{31}^{ABC}(\varsigma,\tau)=\alpha\tau^2(h[\alpha+M(\alpha)-1][3\alpha+M(\alpha)-3]+M(\alpha)[2\alpha+M(\alpha)-2])$$

$$\begin{aligned}&\psi_{32}^{ABC}(\varsigma,\tau)=\Gamma(2\alpha+1)\Gamma(3\alpha+1)[-(\alpha-1)\Gamma(\alpha+1)\\&\times[\alpha+M(\alpha)-1](h[\alpha+M(\alpha)-1]+M(\alpha))]\end{aligned} \tag{36}$$

and

$$\psi_{33}^{ABC}(\varsigma,\tau)=\alpha^3\Gamma(\alpha)\tau^{2\alpha}(\alpha h\Gamma(2\alpha+1)\tau^\alpha-\Gamma(3\alpha+1)[h3\alpha+2M(\alpha)-3]+M(\alpha)]). \tag{37}$$

Following the same procedure, we can obtain the remaining approximations. We, therefore, have the following approximate solution of the equations (3) to (5), which we have derived here by using the HATM:

$$\psi^{(\cdot)}(\varsigma,\tau)=\psi_0^{(\cdot)}(\varsigma,\tau)+\sum_{j=1}^{m}\psi_0^{(\cdot)}(\varsigma,\tau), \tag{38}$$

where the superscript $(\cdot)$ is to be replaced by (LC), (CFC), (ABC).

3. Graphical Illustrations and Numerical Results

In Figure 1 (a) to Figure 1 (c), we compare the HATM solutions of TFGDE by using LC, CFC and ABC operators for $\alpha=0.3, 0.5, 0.99,$ respectively, with $\tau=5$ and $h=-1.5.$ From Figure 1 we observe that the solutions have the same behavior and are close to each other. For α close to 1, the solutions coincide with each other.

In order to determine the effect of the value of h on the convergence of the HATM solutions, we plot the h curves of $\psi_\tau(\varsigma,0)$ given by the sixth-order HATM solution (38) against h. The interval of convergence is the region of the h-curves which is parallel to the h-axes.

Figures 2 to 4 show the h-curves for (3) to (5) for different values of α. It is clear from each of these figures that the h-curves are identical when the order α approaches to 1.

Table 1 gives the absolute error when we use the first 6 terms of the HATM solution (38) for the operators LC,CFC and ABC in comparison with the exact solution of the classical gas dynamics equation. The order of these errors ranges form 10^{-3} to 10^{-8}.

The above-mentioned behavior shows the effectiveness and accuracy of the approximate solutions for the operators ${}^{CFC}_{0}D^{\alpha}_{\tau}\psi$, ${}^{ABC}_{0}D^{\alpha}_{\tau}\psi$. We can also approximately find the region of convergence in h for the HATM solution (38).

4. Concluding remarks and observations

In the paper, the HATM was utilized to evaluate the approximate solutions of the time-fractional gas dynamics equation (TFGDE). By using the Liouville-Caputo (LC), Caputo-Fabrizio (CFC) and Atangana-Baleanu (ABC) time–fractional derivative operators, we presented alternative solutions to TFGDE. We compared these approximate solutions with each other and a remarkably good agreement was found. Also, the interval of convergence for the Homotopy Analysis Transform Method (HATM) and the comparison with the exact solution for $\alpha = 0.99$ were computed by means of Mathematica and a good agreement was found again. The behavior of the approximation that have been calculated by HATM based upon the LC, CFC and ABC fractional-order derivative operators was indicated to show the accuracy and effectiveness of this method for each of these fractional-order derivative operators. In the case when the fractional-order derivative became the classical derivative, the approximate solutions derived in this paper coincided with those given in earlier works.

Table 1. The absolute error for the 6 terms of the HATM solution (2.36) by using the operators LC, CFC and ABC in comparison with the exact solution of the classical gas dynamics for $\alpha = 0.99,\ \varsigma = 10$ and $\alpha = 1$.

τ	Error for LC	Error for CF	Error for ABC
0.00	1.03530×10^{-8}	1.39257×10^{-7}	1.3925691×10^{-7}
0.50	5.08056×10^{-7}	9.80808×10^{-7}	1.50003×10^{-6}
1.00	9.78829×10^{-7}	1.88506×10^{-6}	2.9608×10^{-6}
1.50	9.08386×10^{-7}	2.7446×10^{-6}	4.59977×10^{-6}
2.00	2.74971×10^{-6}	7.97116×10^{-7}	3.65658×10^{-6}
2.50	1.93700×10^{-5}	1.28798×10^{-5}	8.95815×10^{-6}
3.00	7.19327×10^{-5}	6.05798×10^{-5}	5.5972×10^{-5}
3.50	2.09898×10^{-4}	1.90757×10^{-4}	1.86651×10^{-4}
4.00	5.31289×10^{-4}	5.00016×10^{-4}	4.98911×10^{-4}
4.50	1.21995×10^{-3}	1.17026×10^{-3}	1.17661×10^{-4}
5.00	2.60782×10^{-3}	2.53086×10^{-3}	2.55183×10^{-3}

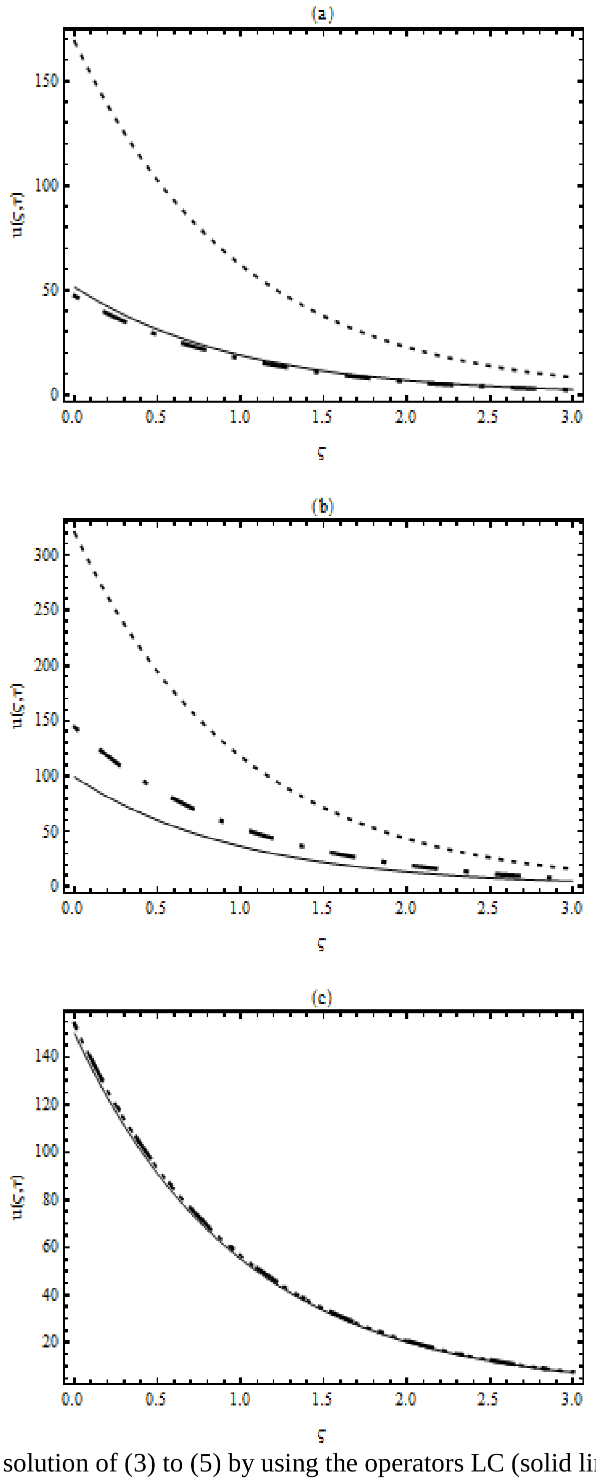

Figure 1. The HATM solution of (3) to (5) by using the operators LC (solid line), CFC (dashed line) and ABC (dashed-dotted line), respectively, for $\tau = 5$, $h = -1.5$ and $\alpha = 1$: (a) $\alpha = 0.3$; (b) $\alpha = 0.5$; (c) $\alpha = 0.99$.

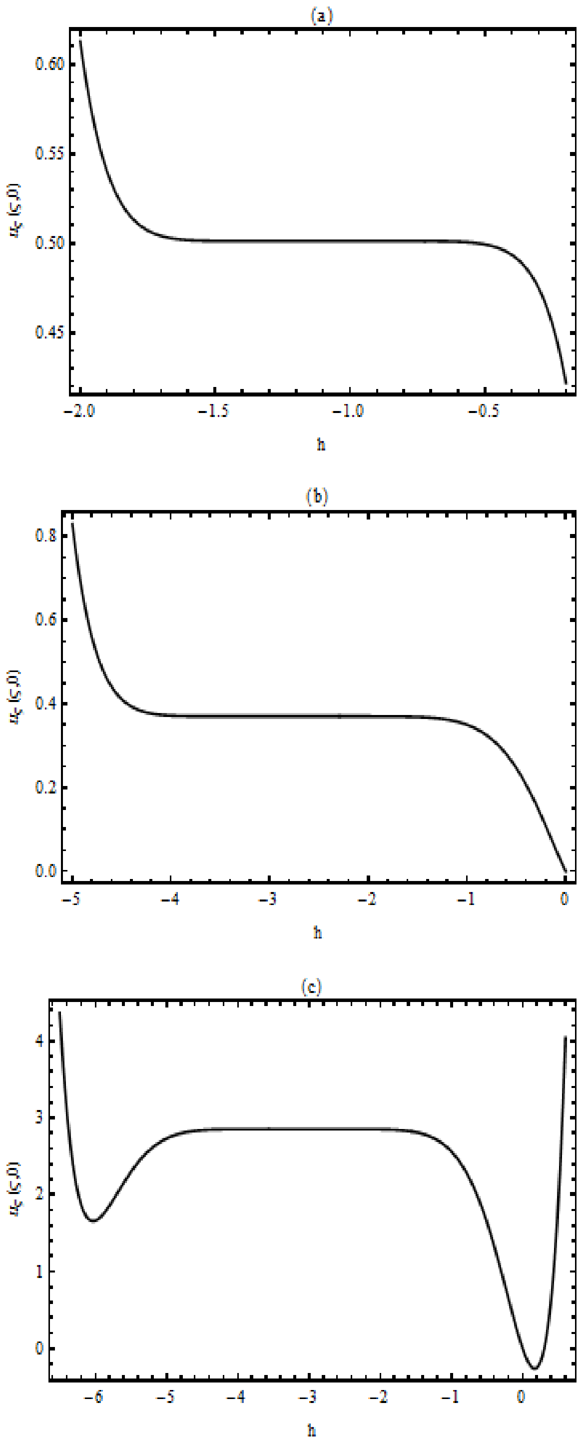

Figure 2. The h-curves of $\psi_{\tau}(\varsigma,0)$ at the 6th term of the HATM solutions by using the operators LC, CFC and ABC for $\varsigma = 0.3, \alpha = 0.5$ and $\alpha = 0.1$: (a) LC; (b) CFC; (c) ABC.

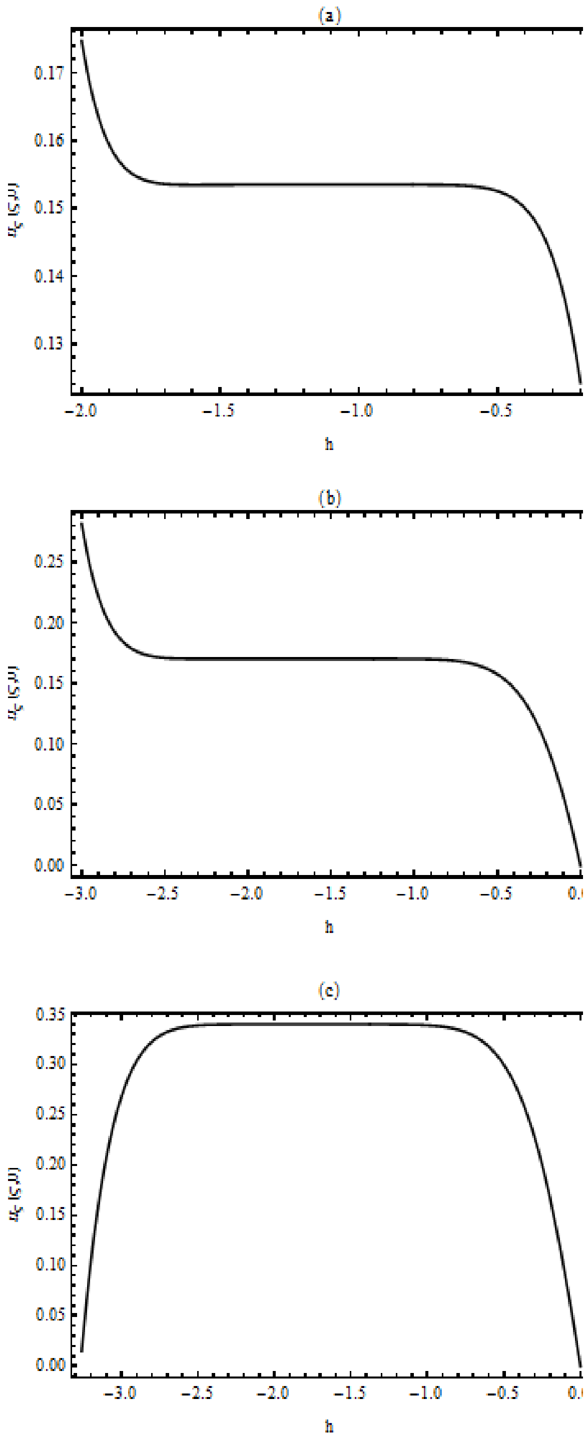

Figure 3. The h-curves of $\psi_{\tau}(\varsigma,0)$ at the 6th term of the HATM solutions by using the operators LC, CFC and ABC for $\varsigma = 0.3, \alpha = 0.7$ and $\alpha = 0.1$: (a) LC; (b) CFC; (c) ABC.

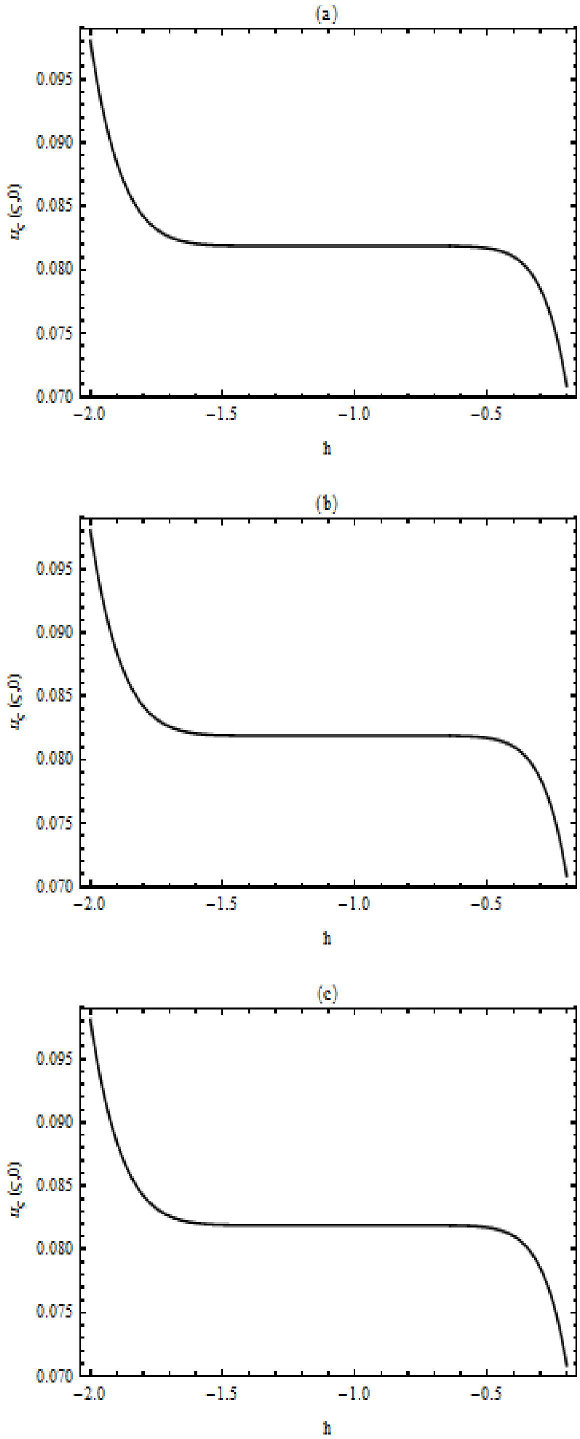

Figure 4. The h-curves of $\psi_\tau(\varsigma,0)$ at the 6th term of the HATM solutions by using the operators LC, CFC and ABC for $\varsigma=0.3$, $\alpha=0.9999$ and $\alpha=0.1$: (a) LC; (b) CFC;+ (c) ABC.

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